Does a system of linear equations have solutions?

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Learning Target

Review

System of Linear Equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Matrix-vector product: $A\mathbf{x} = \mathbf{b}$

- Given A and b, sometimes x exists (having solution), and sometimes doesn't (no solution)
- New terms: "*linear combination*" and "*span*"

Solution

- Given a system of linear equation

$$
\begin{array}{rcl}
 & & \vdots \\
2x_1 - 3x_2 + x_3 & = & -10 \\
x_1 \qquad\quad + x_3 & = & 3
\end{array}
$$

$\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ is a solution

$\begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}$ is also a solution

There are other possible solutions ……

The **set** of **all solutions** of a **system of linear equations** is called the **solution set**.

Solution

- A system of linear equations is called consistent if it has one or more solutions.
- A system of linear equations is called inconsistent if its solution set is empty.

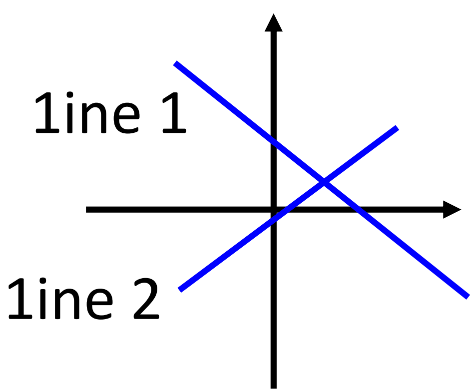
	$\begin{aligned} 3x_1 + x_2 &= 10 \\ x_1 - 3x_2 &= 0 \end{aligned}$	$\begin{aligned} 3x_1 + x_2 &= 10 \\ 6x_1 + 2x_2 &= 20 \end{aligned}$	$\begin{aligned} 3x_1 + x_2 &= 10 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$
Solution set	$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \end{bmatrix} : \forall t \in \mathcal{R} \right\}$	$\{\}$, or ϕ
Consistent or Inconsistent?	Consistent	Consistent	Inconsistent

Solution

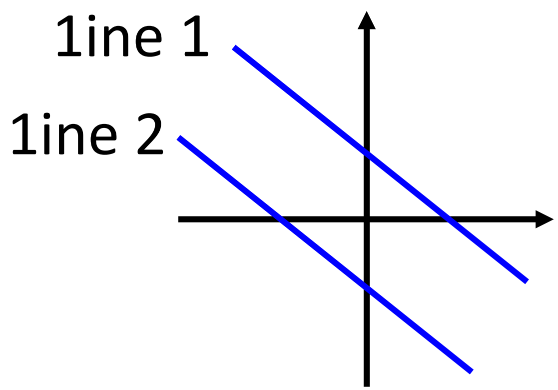
- Considering any system of linear equations with 2 variables and 2 equations

$$\color{blue}{a_{11}}\color{red}{x_1} + \color{blue}{a_{12}}\color{red}{x_2} = \color{green}{b_1} \quad \text{...... line 1}$$
$$\color{blue}{a_{21}}\color{red}{x_1} + \color{blue}{a_{22}}\color{red}{x_2} = \color{green}{b_2} \quad \text{...... line 2}$$

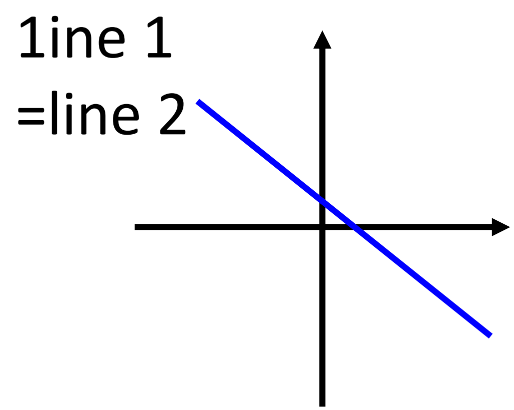
<u>Row Aspect</u>



unique solution



no solution



infinitely many solution

Solution

- Considering any system of linear equations with 2 variables and 2 equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \text{...... line 1}$$
$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \text{...... line 2}$$

- How about 3 variables and 2 equations?

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

- How about 3 variables and 3 equations?

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

More Variables?

Does a system of linear equations have solutions?

Linear Combination

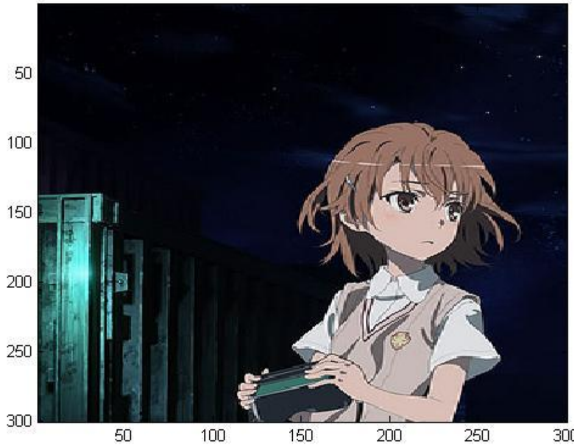
Linear Combination

- Given a vector set $\{u_1, u_2, \cdots, u_k\}$
- The linear combination of the vectors in the set:
  - $v = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$
  - $c_1, c_2, \cdots, c_k$ are scalars (Coefficients of linear combination)

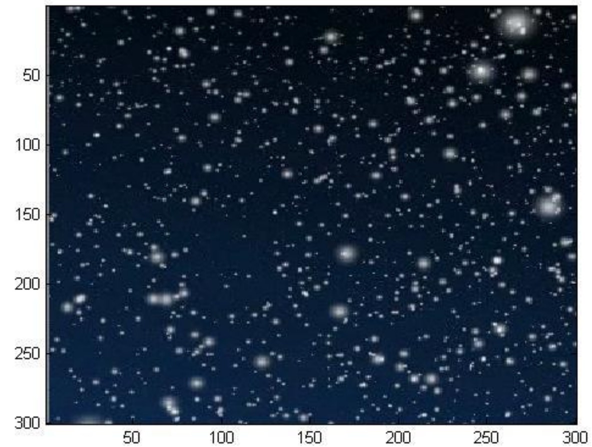
vector set: $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

coefficients: $\{-3,4,1\}$

What is the result of linear combination?

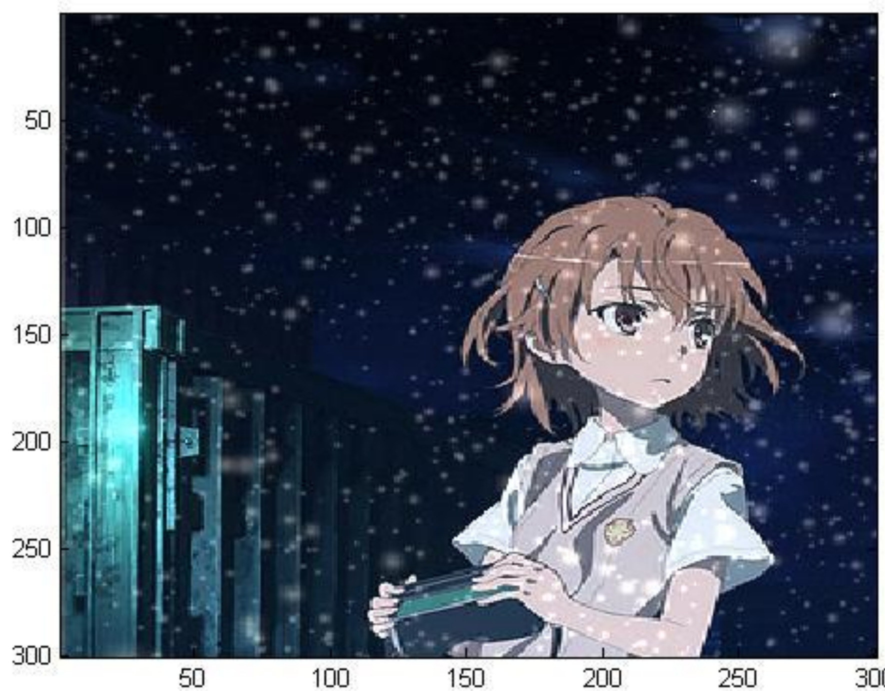


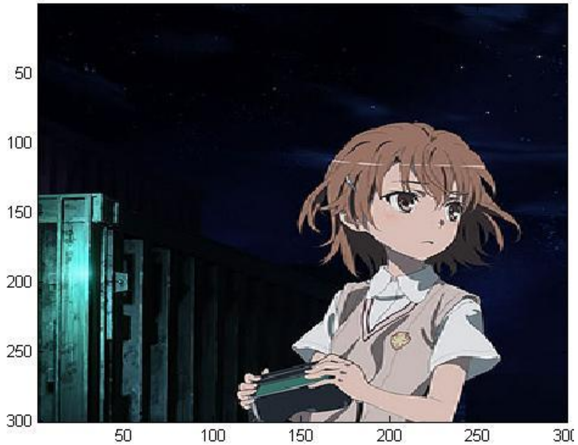
u_1



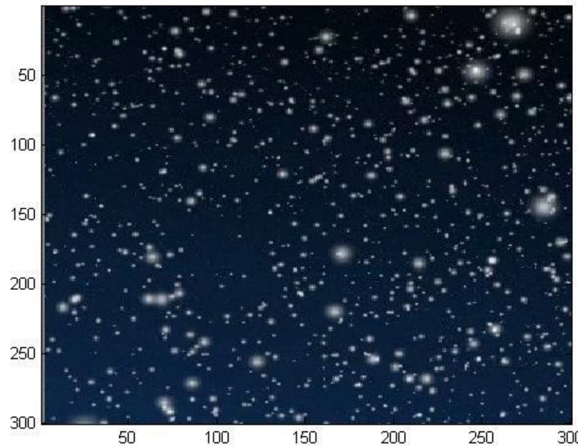
u_2

$$1 \cdot u_1 + 0.5 \cdot u_2 =$$



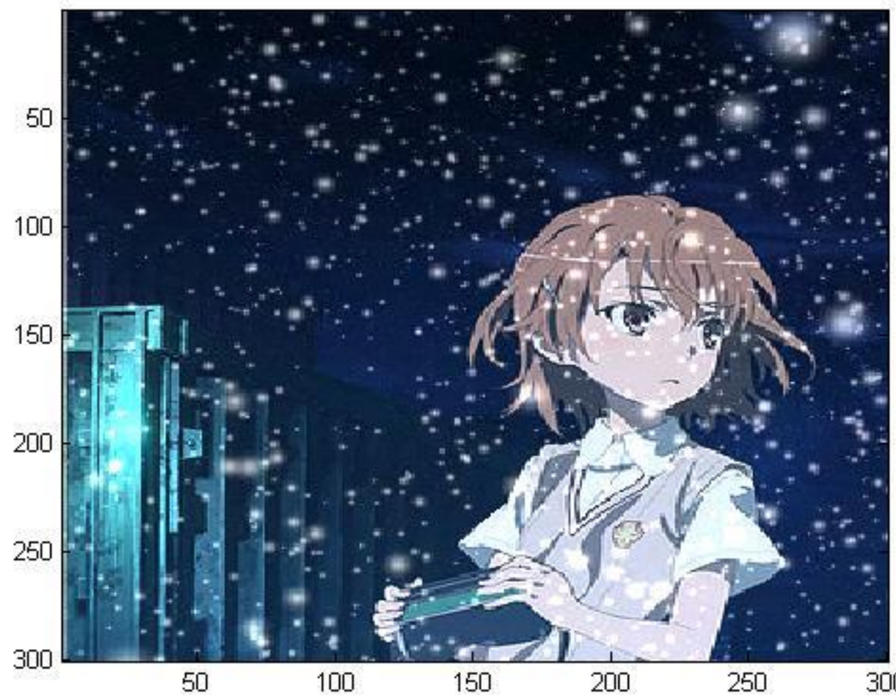


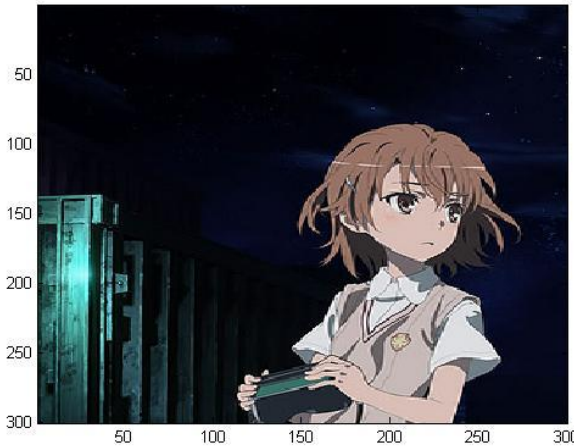
u_1



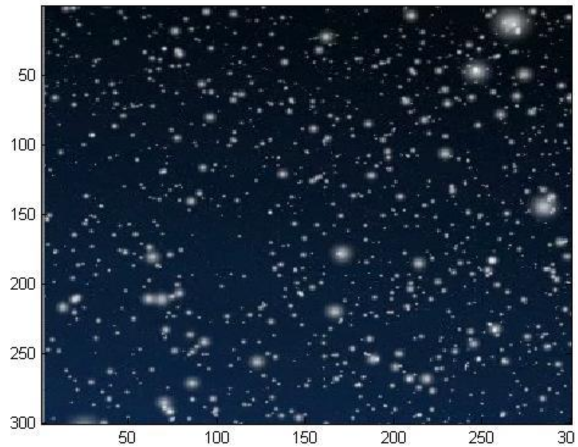
u_2

$$1 \cdot u_1 + 1 \cdot u_2 =$$

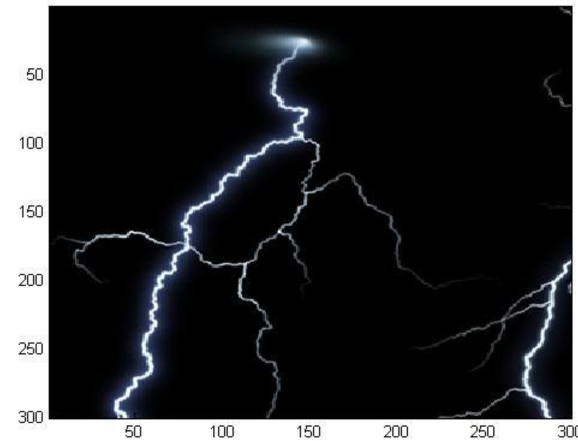




u_1

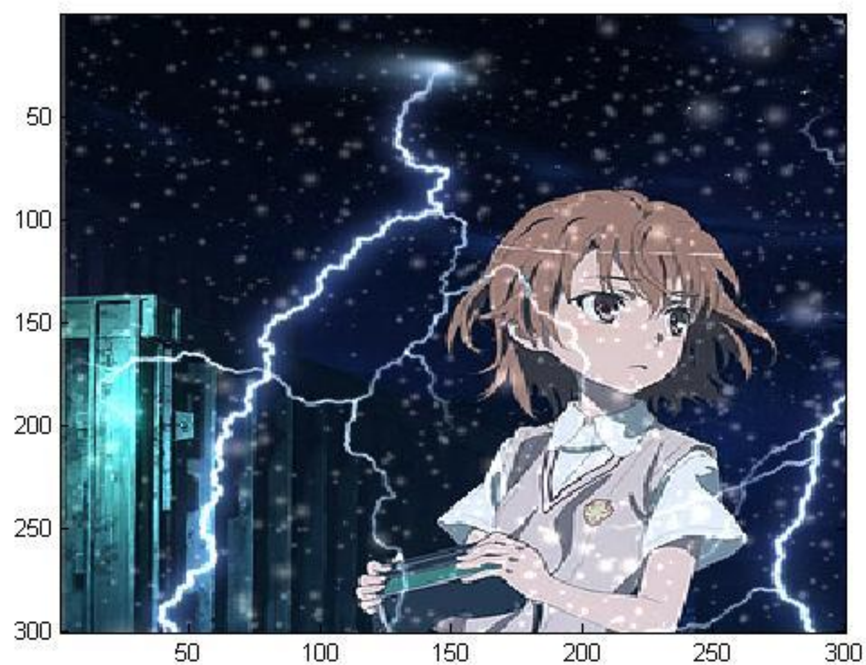


u_2



u_3

$$1 \cdot u_1 + 0.5 \cdot u_2 + 1 \cdot u_3 =$$



Column Aspect

$$\begin{array}{ccccccccc} \boxed{a_{11}} x_1 & + & \boxed{a_{12}} x_2 & + & \cdots & + & \boxed{a_{1n}} x_n & = & b_1 \\ a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & b_m \\ a_1 & & a_2 & & & & a_n & & \end{array}$$

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

Vector set

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

coefficients

$$\mathbf{Ax} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Linear Combination

System of Linear Equations v.s. Linear Combination

Non empty solution set?

Has solution or not?

Consistent?

$$A\mathbf{x} = \mathbf{b}$$

(A system of linear equations)

The Same question

Column Aspect

$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$$

the linear combination of columns of A

Is b the linear combination of columns of A?

Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Is b the linear combination of columns of A?

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$$

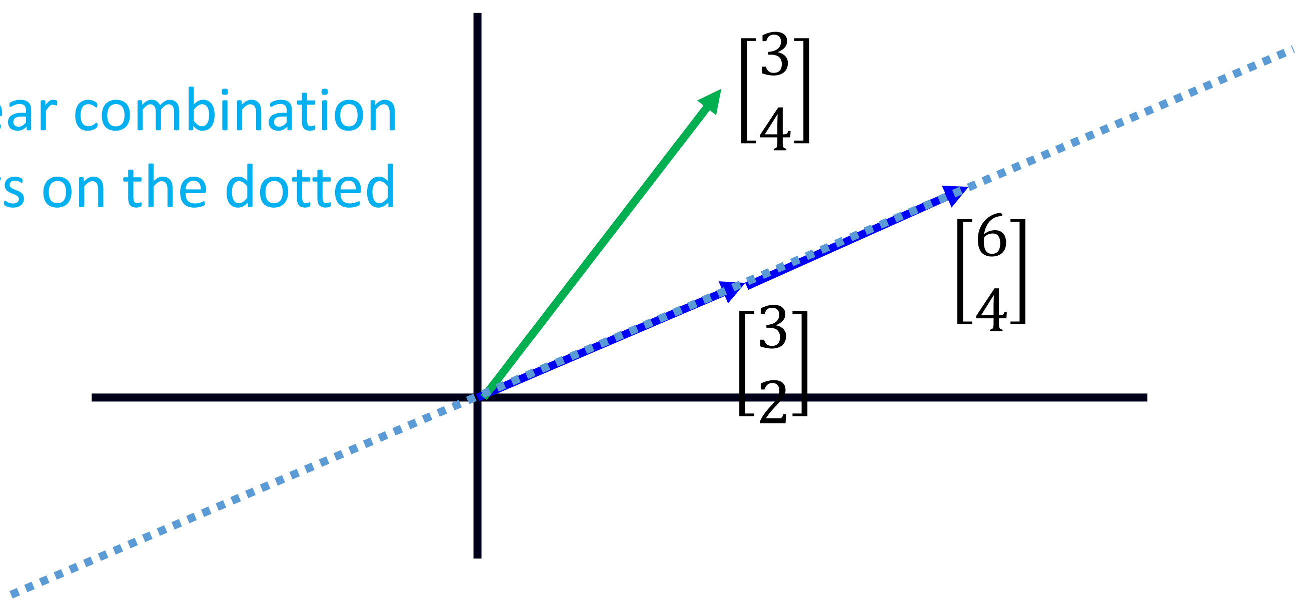
Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

- Vector set: $\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}\right\}$
- Is $\begin{bmatrix}3\\4\end{bmatrix}$ a linear combination of $\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}\right\}$? No

The linear combination is always on the dotted line.



Example 2

$$2x_1 + 3x_2 = 4$$
$$3x_1 + 1x_2 = -1$$

Has solution or not?

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Is b the linear combination of columns of A?

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

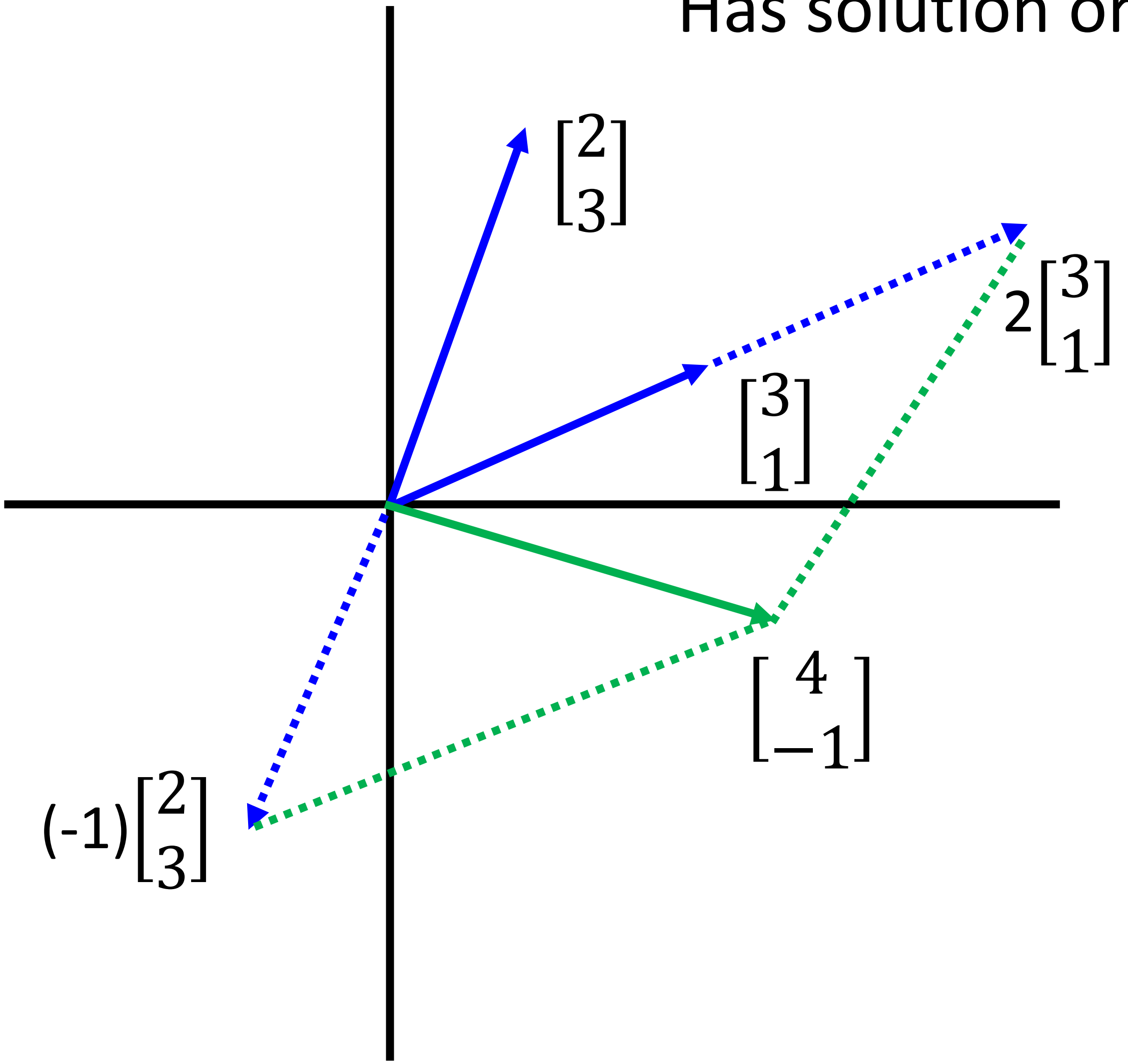
Example 2

$$2x_1 + 3x_2 = 4$$
$$3x_1 + 1x_2 = -1$$

Has solution or not?

$$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

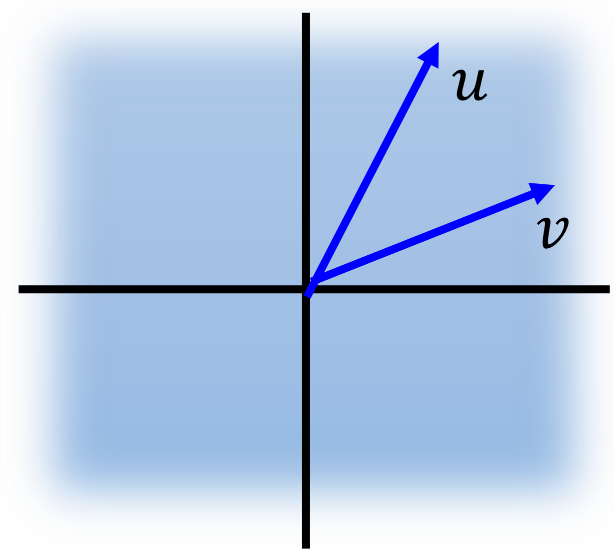
$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$



Example 2

How about in $\mathcal{R}^4$?

- If **u** and **v** are any nonparallel vectors in $\mathcal{R}^2$, then every vector in $\mathcal{R}^2$ is a linear combination of **u** and **v**
  - Nonparallel: **u** and **v** are nonzero vectors, and **u** $\neq c$**v**.



$$\begin{aligned} u_1x_1 \quad &+v_1x_2 \quad &= b_1 \\ u_2x_1 \quad &+v_2x_2 \quad &= b_2 \end{aligned}$$

u and **v** are not parallel

?

Has solution

- If **u, v** and **w** are any nonparallel vectors in $\mathcal{R}^3$, then every vector in $\mathcal{R}^3$ is a linear combination of **u, v** and **w**?

Example 3

$$2x_1 + 6x_2 = -4$$
$$1x_1 + 3x_2 = -2$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Is b the linear combination of columns of A?

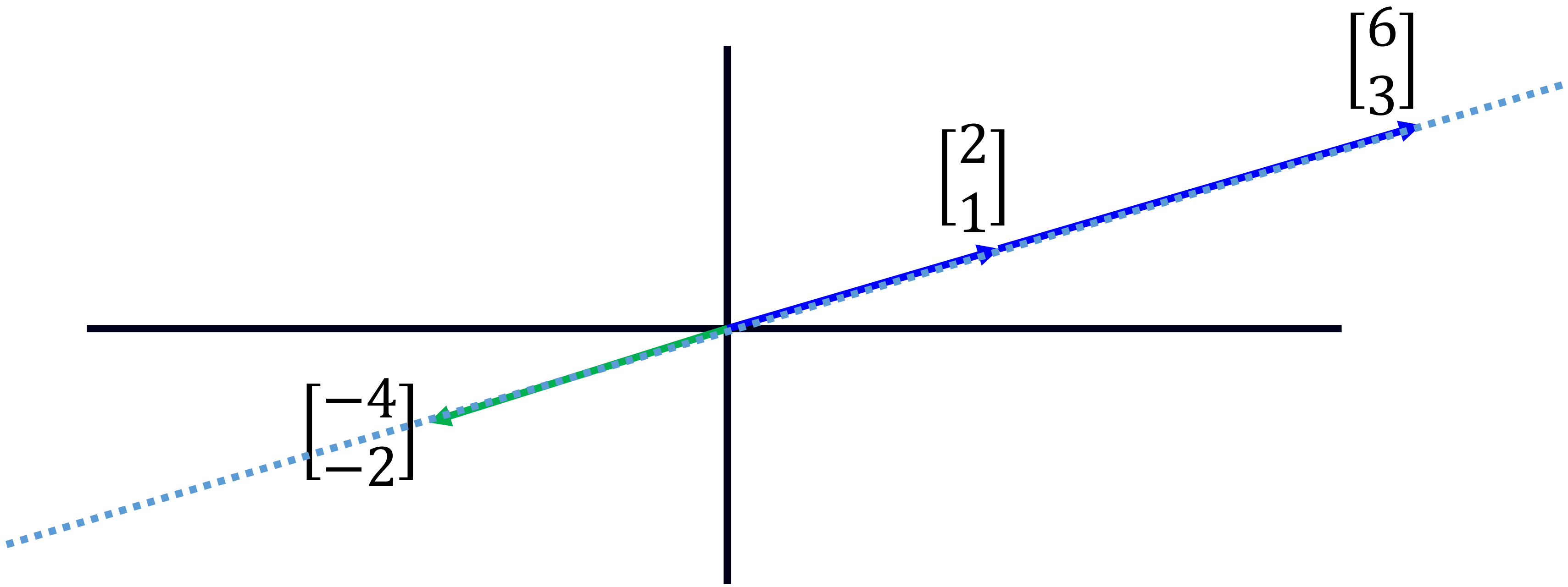
$$\begin{bmatrix} -4 \\ -2 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$$

Example 3

$$2x_1 + 6x_2 = -4$$
$$1x_1 + 3x_2 = -2$$

Has solution or not?

- Vector set: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$
- Is $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ a linear combination of $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$? Yes



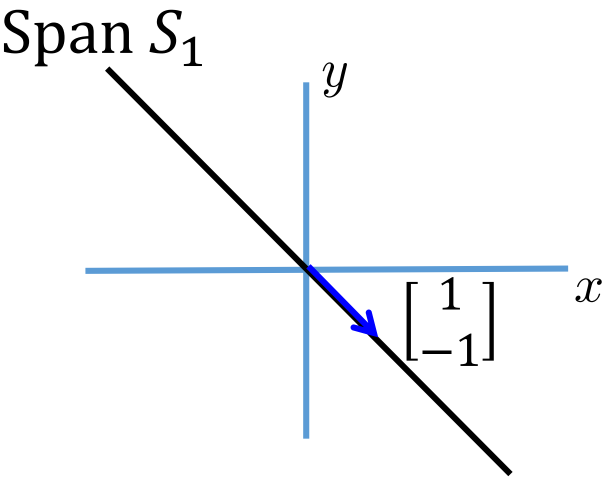
Does a system of linear equations have solutions?

Span

Span

- A vector set $S = \{u_1, u_2, \cdots, u_k\}$
- Span of S is the vector set of all linear combinations of $u_1, u_2, \cdots, u_k$
  - Denoted by $Span\ \{u_1, u_2, \cdots, u_k\}$ or $Span\ S$
  - $Span\ S = \{c_1 u_1 + c_2 u_2 + \cdots +$

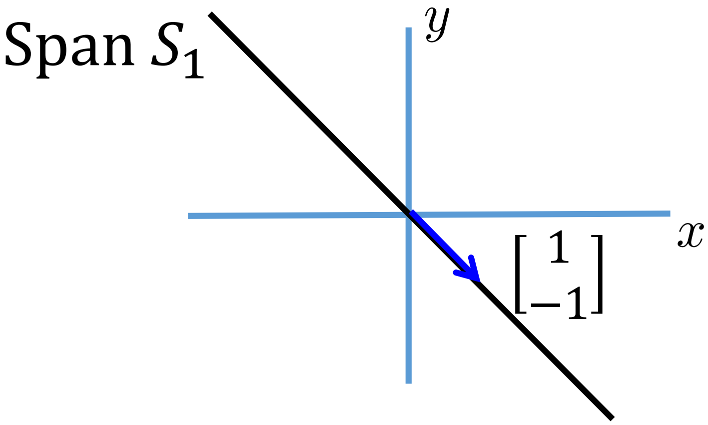
Span

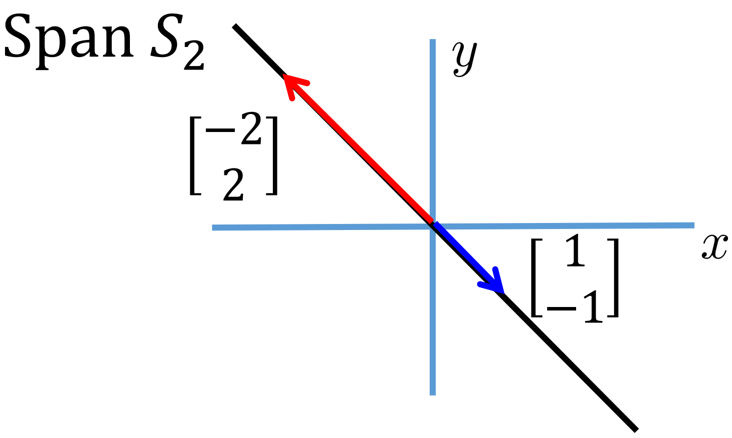


- Let $S_0 = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$, what is Span S_0?
  - Ans: $\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$ (only one member)
- Let $S_1 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$, what is Span S_1?
  - If S contains a non zero vector, then Span S has infinitely many vectors

Span

- Let $S_1 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$, what is Span S_1?
- Let $S_2 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\}$, what is Span S_2?





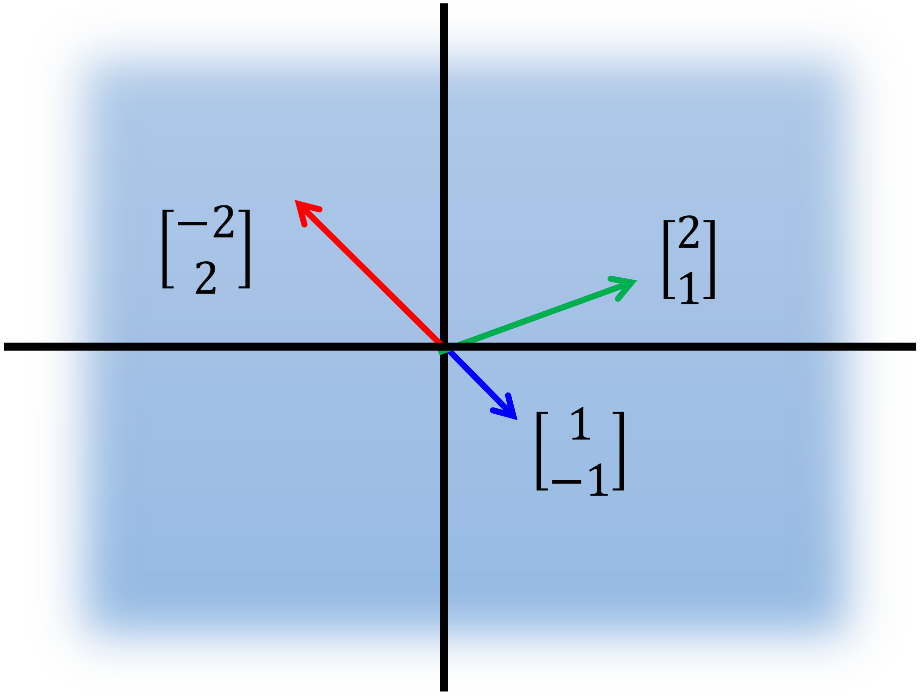
$$\text{Span } S_1 = \text{Span } S_2$$

(Different number of vectors can generate the same space.)

Span

- Let $S_3 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$, what is Span S_3?

nonparallel vectors

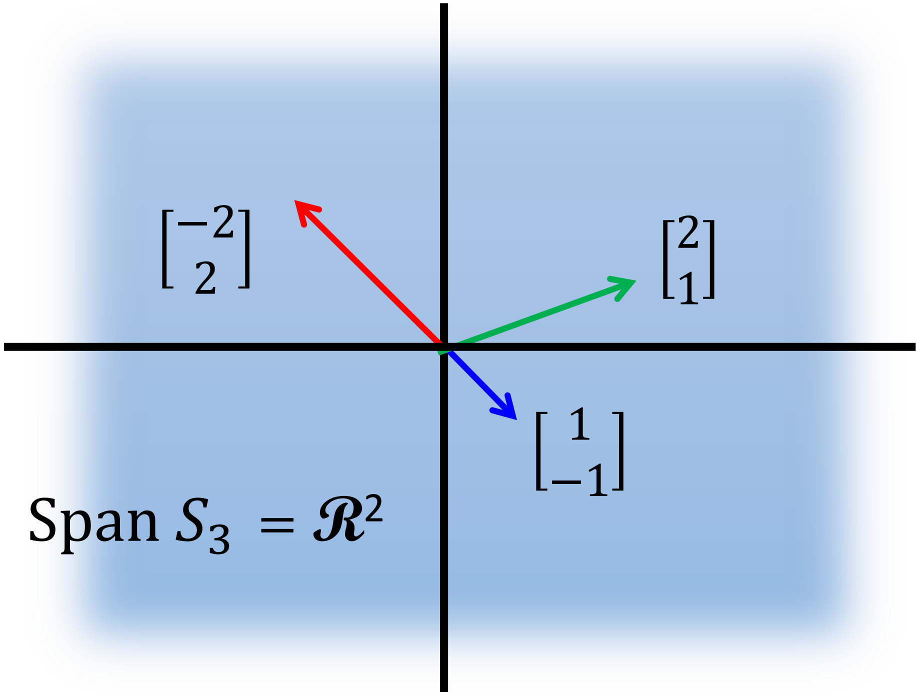


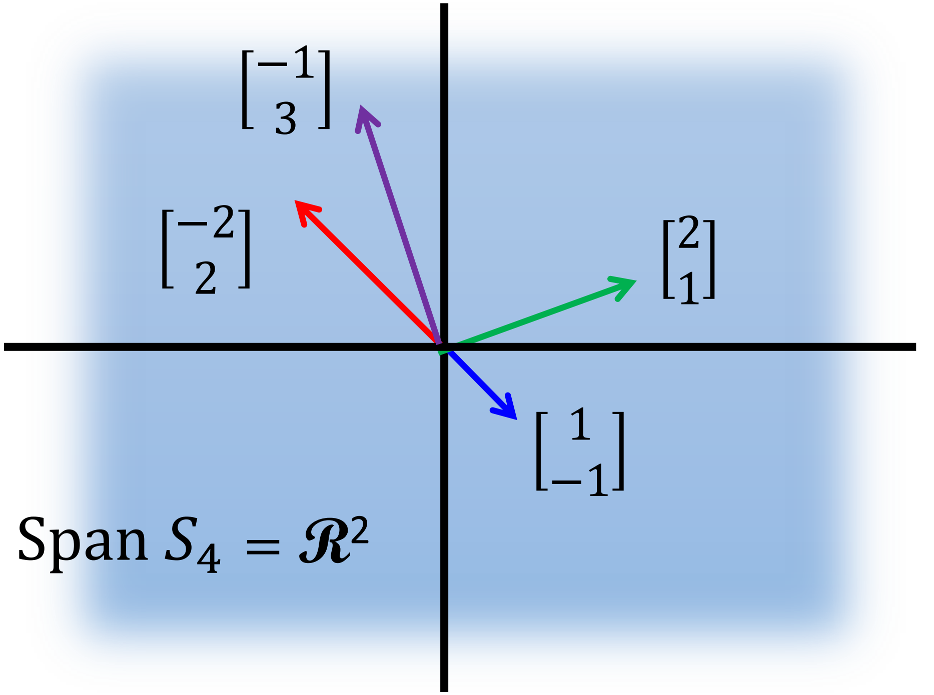
Every vector in $\mathcal{R}^2$ is their linear combination

$$\text{Span } S_3 = \mathcal{R}^2$$

Span

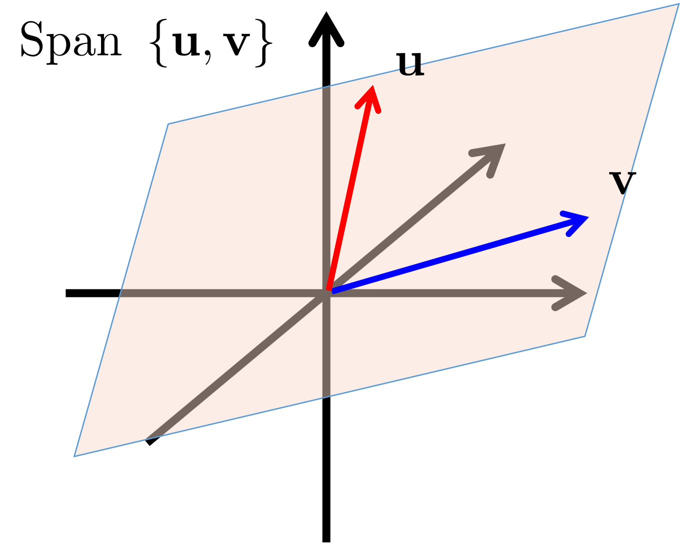
- Let $S_3 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$, what is Span S_3?
- Let $S_4 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\}$, what is Span S_4?

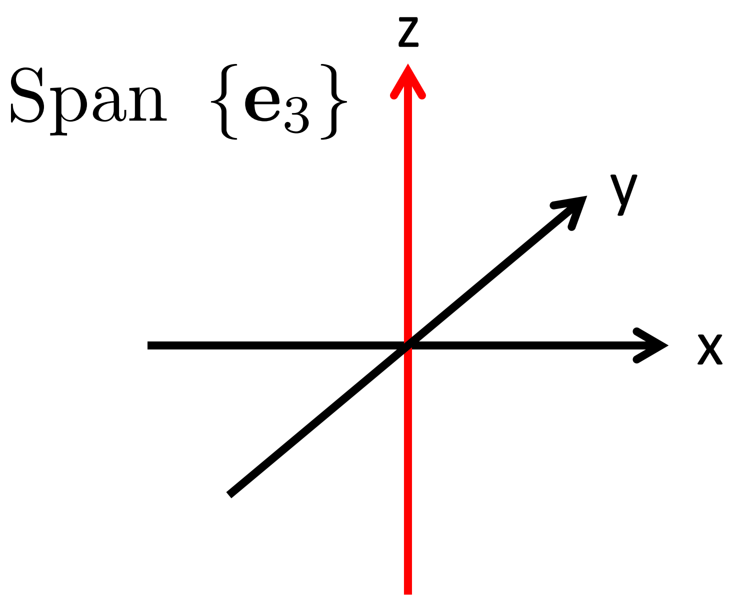


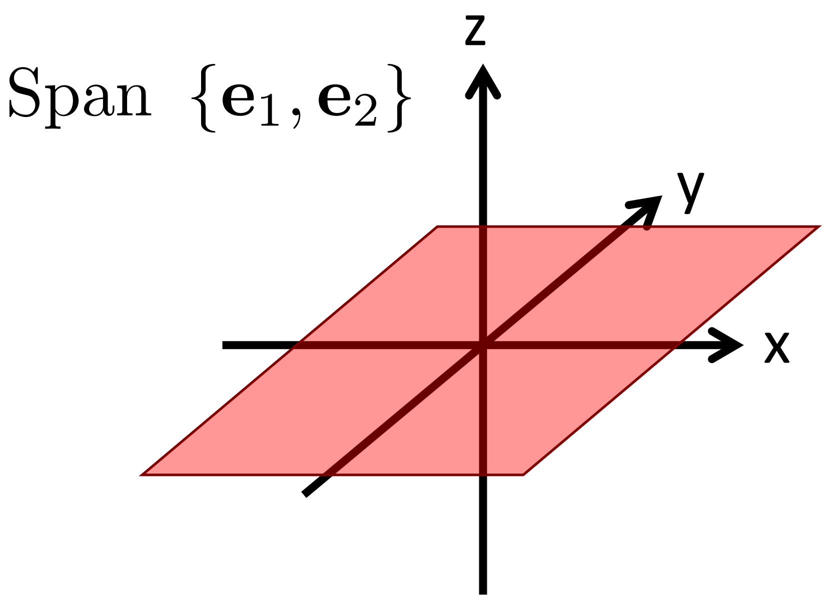


Span

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$







$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$A\mathbf{x} = \mathbf{b}$$

Has solution or not?

The same question

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Is b the linear combination of columns of A?

The same question

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in Span \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \ldots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

Is b in the span of the columns of A?

Summary

$$A\mathbf{x} = \mathbf{b}$$

Does a system of linear equations have solution?

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

YES $\Rightarrow$ Have solution

NO $\Rightarrow$ No solution

Acknowledgement

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Eigenface